Q4. Discuss the linear convolution of the 2-finite length sequences using the DFT

DTFT: A Quick Recap

- Extends the FT for non-periodic discrete-time signals
- Forward DTFT:

$$X[\Omega] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

- Periodic spectrum of period 2π.
- Abandon to use the DTFT in a digital signal processer for the following reasons:
  - DTFT Spectrum $X[\Omega]$ is continuous
  - Real signals have finite length

Discrete Fourier Transform

- Extends the DTFT for non-periodic discrete-time signals (finite duration) with discrete frequencies.
- Samples the DTFT spectrum on the interval $[0, 2\pi]$ using N points.
- N-point DFT-pairs:
  - Forward

$$X[k] \;=\; \sum_{n=0}^{N-1} x[n]W^{kn}, \quad k = 0, \ldots (N-1)$$

where

$$W = \exp(-j\frac{2\pi}{N}).$$

  - Inverse

$$x[n] \;=\; \sum_{k=0}^{N-1} X[k]W^{-kn}, \quad n = 0, \ldots (N-1).$$

DFT-pairs in Block-Matrix Form

- Let

$$
\begin{aligned}
\mathbf{x} &= [x(0), x(1), \ldots x(N-1)]^T \\
\mathbf{X} &= [X(0), X(1), \ldots X(N-1)]^T \\
\mathbf{W} &= \begin{pmatrix} W^0 & W^0 & \ldots & W^0 \\ W^0 & W^1 & \ldots & W^{N-1} \\ \ldots & \ldots & \ldots & \ldots \\ W^0 & W^{N-1} & \ldots & W^{(N-1)^2} \end{pmatrix}
\end{aligned}
$$

- DFT-pairs in a matrix form:

$$\mathbf{X} = \mathbf{W}\mathbf{x} \tag{1}$$

$$\mathbf{x} = \frac{1}{N}\mathbf{W}^{\mathbf{H}}\mathbf{x} \tag{2}$$

- Requires N^2 complex multiplications and $N(N-1)$ complex additions.

DFT-pairs (Cont'd)

- Taking complex-conjugate of (2) twice replaces the IDFT with DFT:

$$\mathbf{x} \;=\; \frac{1}{N}\big(\mathrm{DFT}(\mathbf{X}^*)\big)^*.$$

- Can be implemented using lightening-speed algorithms!

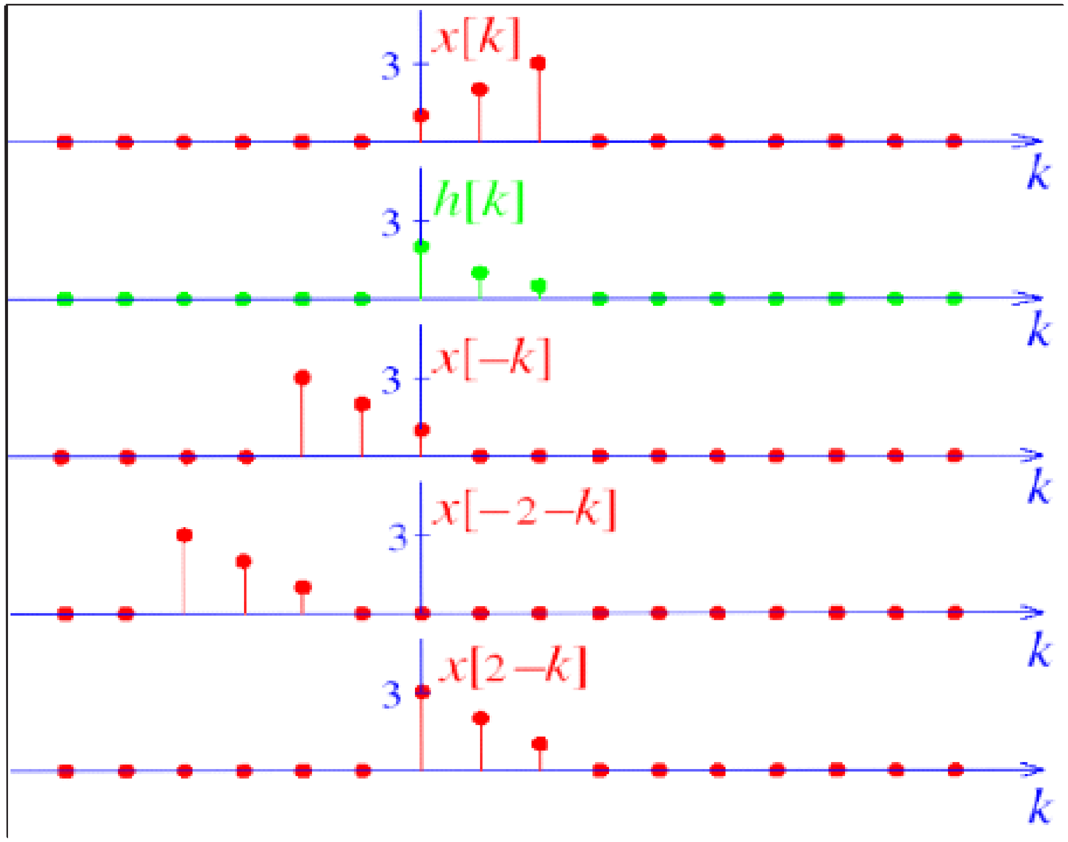
Linear Convolution

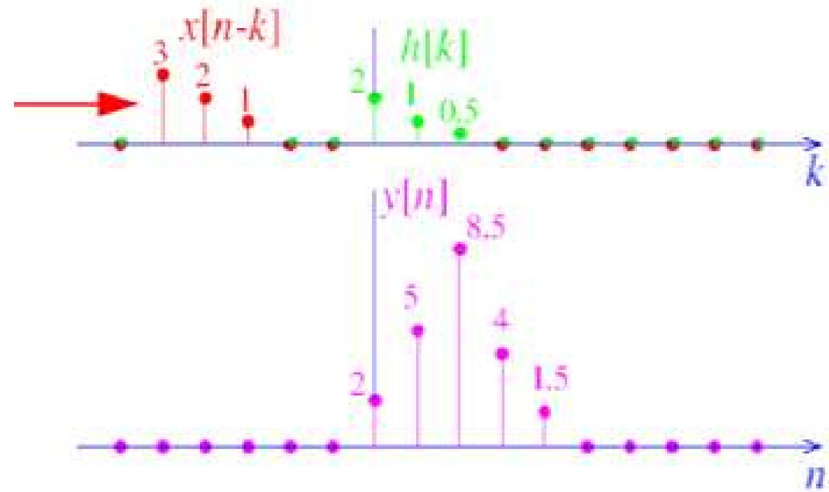
- Definition: Suppose two sequences $h[n]$ and $x[n]$ of length L and P, respectively.

$$y[n] = (h * x)[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \quad (3)$$

- Basic operations:
  - Time invert one of the sequences
  - Slide it from $-\infty$ to ∞
  - When sequences intersect, sum their products
- $y[n]$ is a sequence of length $(L + P - 1)$
- Analogous to computing coefficients of the product of two polynomials.

Example: Linear Convolution





Circular Shift

- Define the circular shift of sequence $x[n]$ of length N as

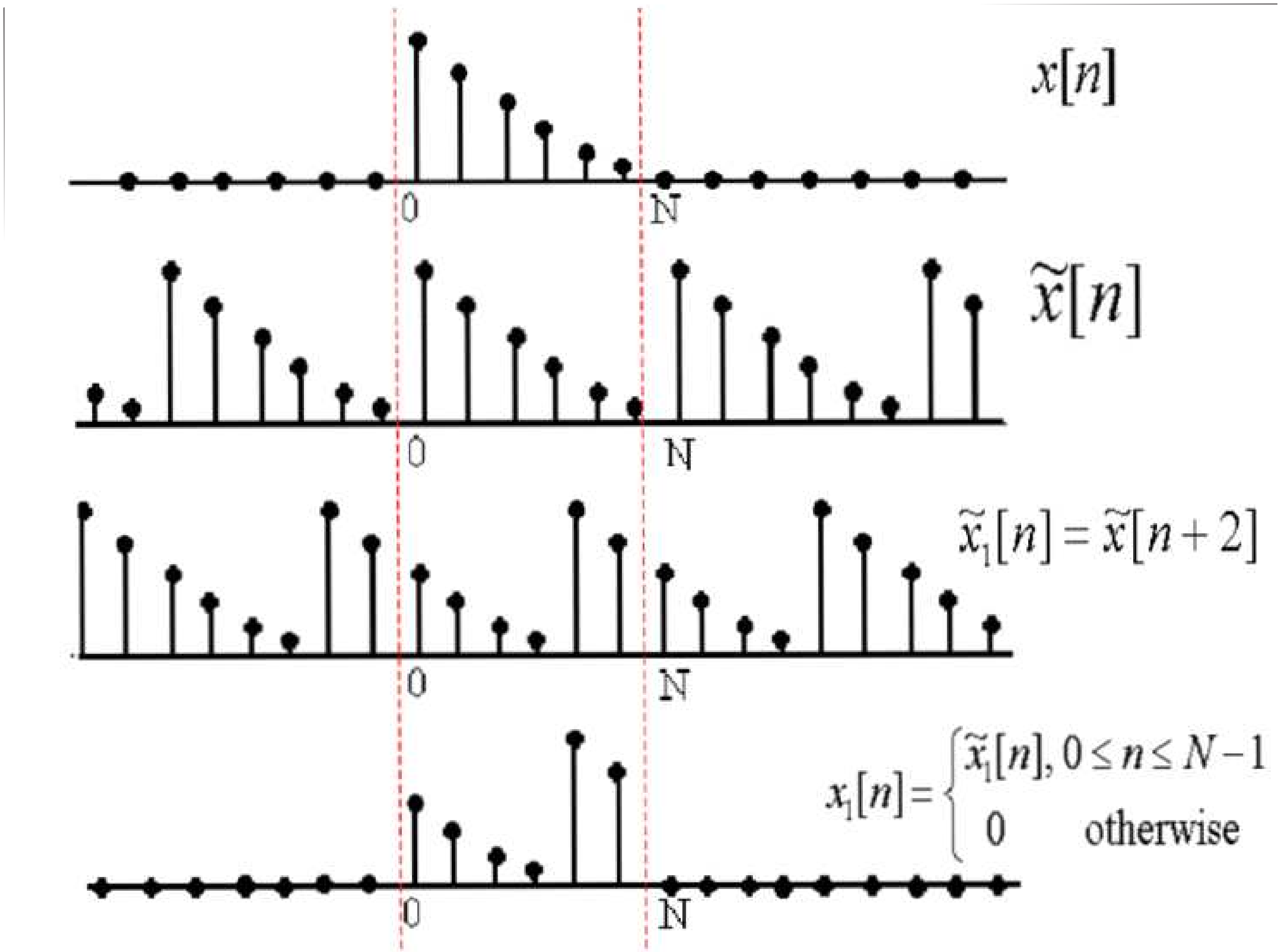
$$x_1[n] \quad = \quad \big(\tilde{x}[m-n]\big)\Pi_N(n)$$

  where

  - $\tilde{x}[n]$ is the periodic extension of $x[n]$
  - $\Pi_N(n)$ the rectangular window in the interval $[0,(N-1)]$.

- 3 basic operations:
  - Periodic extension
  - Normal shift
  - Extraction of the sequence over one period $[0,(N-1)]$

Example (i): Circular Shift



Example (ii): Circular Shift

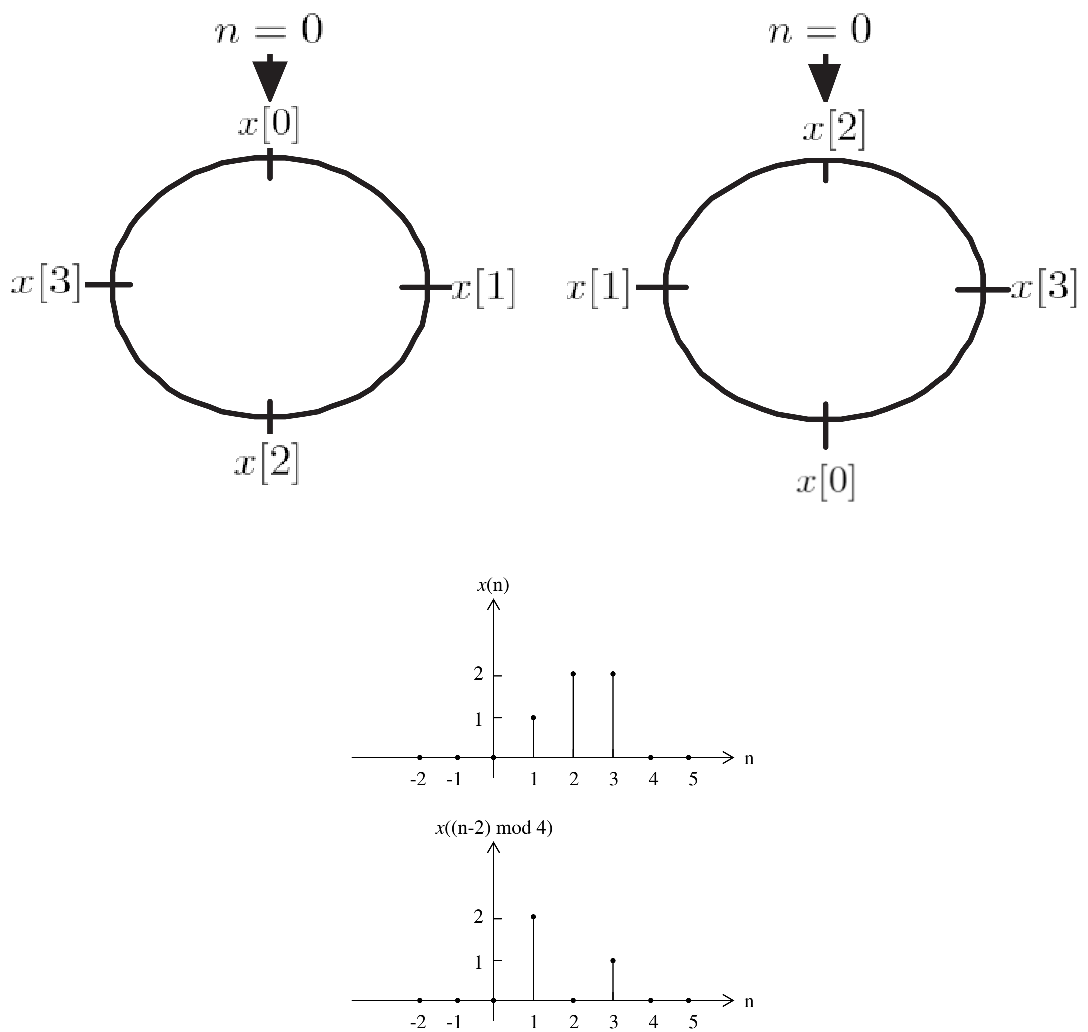


Figure 1: Right Circular Shift on $x[n] = [0, 1, 2, 2]$ by 2 points

Circular Convolution

- Definition: Suppose two sequences $h[n]$ and $x[n]$ of length N each.

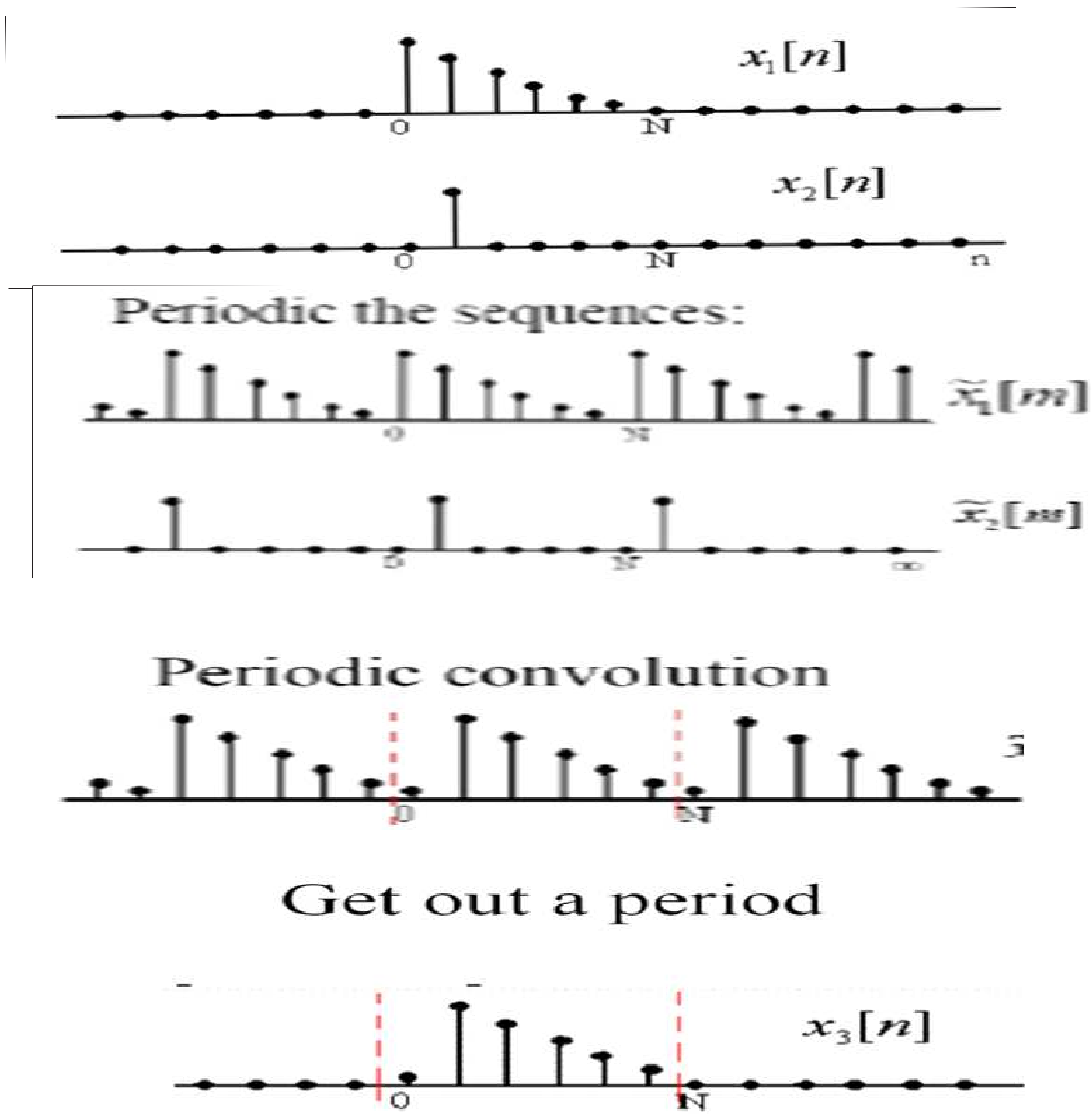
$$y[n] = h[n] \otimes x[n] \quad = \quad \Big(\sum_{m=0}^{N-1} \tilde{h}[m]\tilde{x}[n-m]\Big)\Pi_N(n).$$

- $y[n]$ is a sequence of length N.
- Key Property:

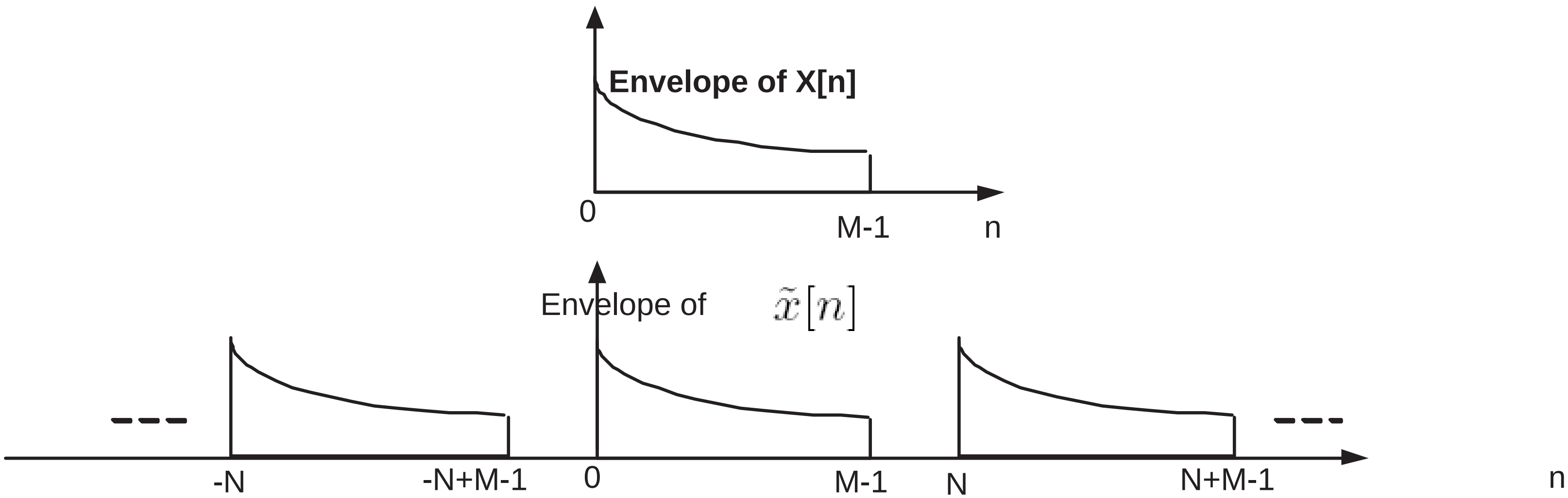
$$h[n] \otimes x[n] \quad \stackrel{\text{DFT}}{\Rightarrow} \quad H[k]X[k]$$

- 3 major differences from the linear convolution:
  - Periodic extension
  - Convolution is confined to one period
  - Truncation of one period at the end

Example: Circular Convolution



Zero Padding



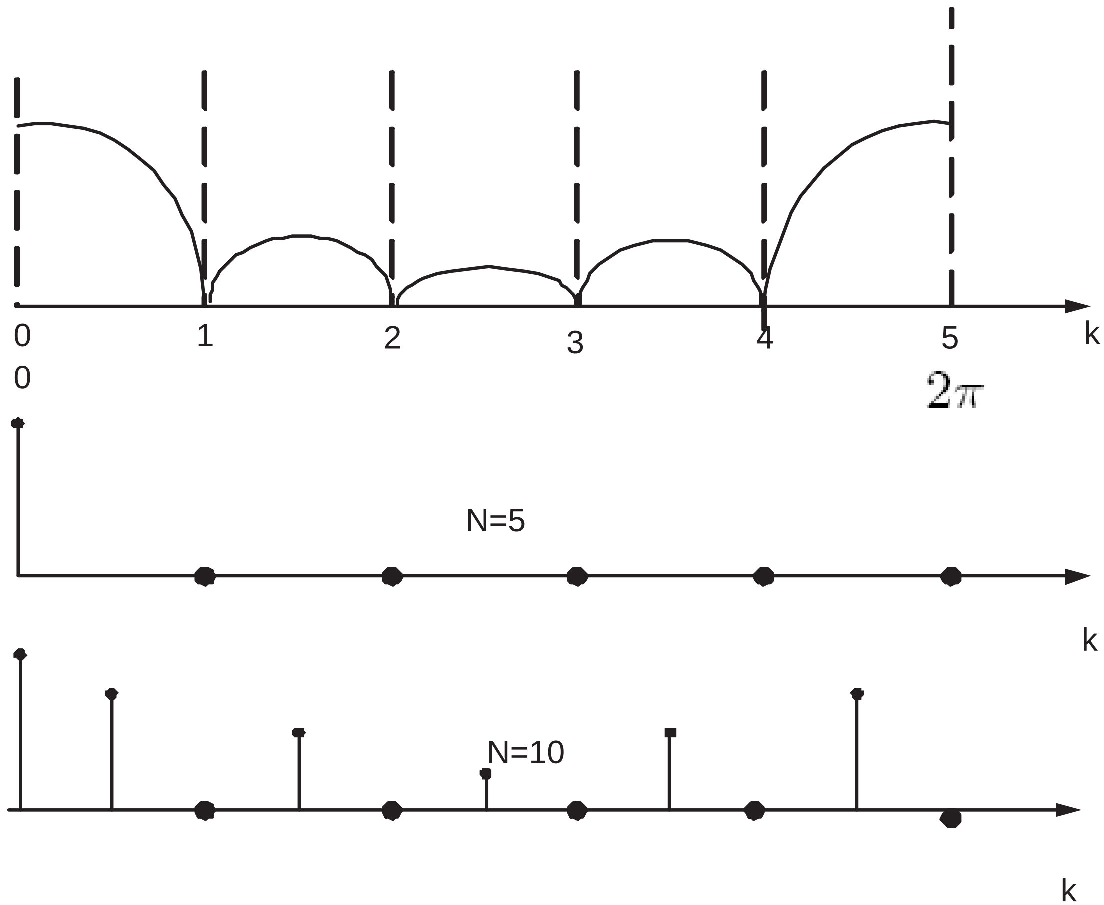
- Can we perform linear convolution using the DFT? If yes, how?
- Extend the length of each sequence such that

$$N \geq M = (L + P - 1),$$

then

$$h[n] \otimes x[n] = h[n] * x[n].$$

Zero Padding (Cont'd)



- Remarks on zero-padding:
  - improves the picture of the DTFT
  - does not increase spectral resolution or reduce the leakage.

Steps: Linear Convolution via DFT

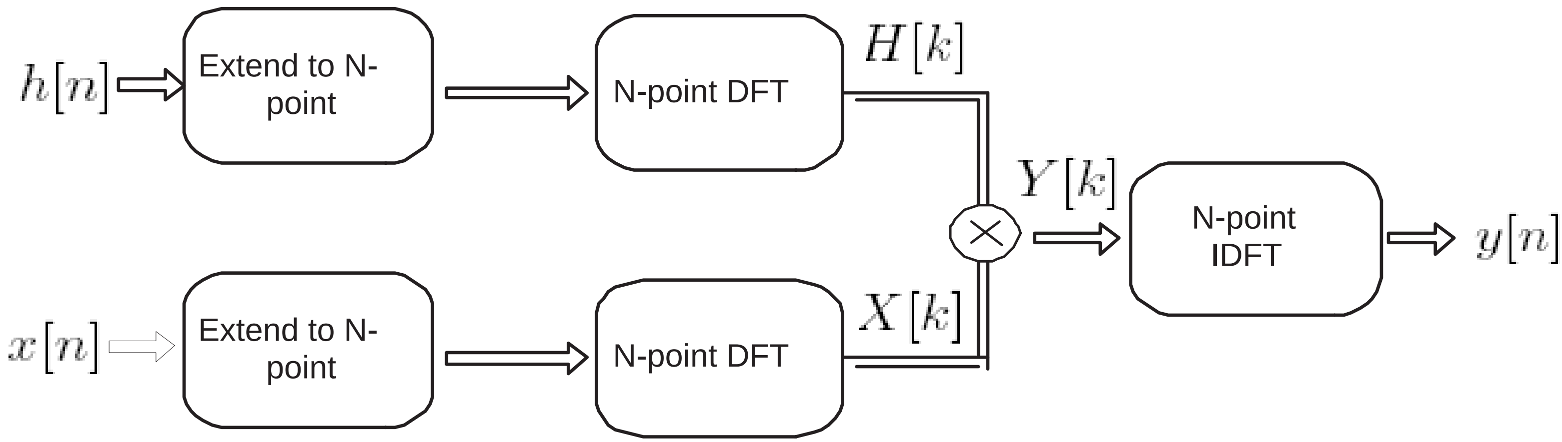


Figure 2: Flow Diagram

- Choose N to be at least $(L + P - 1)$.
- Pad the two original sequences with zeros to length N.
- Compute the N-point DFT to obtain $H[k]$ and $X[k]$.
- Compute the point-wise product:

$$Y[k] \quad = \quad H[k]X[k] \quad k = 0, \ldots (N-1).$$

Linear Convolution (Cont'd)

- Compute $y[n]$ by taking the N-point IDFT of $Y[k]$ as follows:
  - compute the DFT of $Y^*[k]$
  - take the complex conjugate
  - divide by $\frac{1}{N}$
- Save the first $(L+P-1)$ values of $y[n]$.

Final Remarks

- To speed up the process, do the followings:
  1. Use FFT in place of DFT with N being some power of 2.
  2. Suppose $h[n]$ is fixed. So pre-compute and save its DFT in advance.
- Linear convolution via DFT is faster than the 'normal' linear convolution when

$$\underbrace{O(N\log(N))}_{\text{FFT}} < \underbrace{O(LP)}_{\text{normal}}$$

References

- J. K. Zhang, *CoE 4TL4: Digital Signal Processing*, Course notes.
- S. Hayes, *Digital Signal Processing*, Schaum's Outline, 1999.
- A. Oppenheim & R. Schafer *Discrete-time signal processing*, 2nd ed.

Thank you!