Q2. Discuss notion of statistical independence for pair of events, and corresponding situation for multivariate distribution

Events and Probabilities

- The sample space, Ω of an experiment consists of all possible mutually exclusive outcomes .
- An event is a set of outcomes.
- The probability of an event A:

$$P(A) = \lim_{N \to \infty} \frac{\text{No. outcomes of A}, N_A}{\text{No. trials}, N}$$

- Ex. Tossing a fair coin.

$$\Omega = \{H, T\}$$
$$A = \{H\}$$

Hence, $P(A) = \dfrac{1}{2}$

Independent Events

- Definition: Two events A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

- Interpretation:
  - On the LHS, $A \cap B \Rightarrow$ the event that joint/both events A and B occur.
  - On the RHS, we have the product of the probabilities of the individual events/marginals.
  - Intuitively it means that the occurrence of one event does not alter the occurrence probability of the other!

More Insight from the Conditional Probability

- Definition: The conditional probability $P(B|A)$ is the the probability of event B given that A has occurred:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

provided $P(A) \neq 0$.

- If A and B are independent $\Rightarrow$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B).$$

- Interpretation: The event A does not improve our knowledge about the occurrence of B. It makes no difference to B if A has occurred or not.

Toy Example

- Experiment: Toss a coin twice.
- Let A and B be the events of getting head in the first and the second trial, respectively.
- Are the two events independent? Our intuition says Yes !
- Verify from the definition:

$$\begin{aligned} P(A) &= P(HH) + P(HT) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\ P(B) &= P(HH) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\ \text{But, } P(A \cap B) &= P(HH) = \frac{1}{4} = P(A)P(B) \end{aligned}$$

- Hence they are *independent* as expected!

Mutual Exclusiveness and Independence

- Not synonyms!
- If the events A and B are mutually exclusive $\Rightarrow$
  - From the set theory: $A \cap B = \phi$.
  - From the probabilistic point of view:

$$\begin{aligned} P(A \cap B) &= 0 \\ \text{or } P(B|A) &= \frac{P(B \cap A)}{P(A)} = 0 \end{aligned}$$

- Occurrence of A $\Rightarrow$ B has definitely not occurred $\Rightarrow$ A nice piece of information!
- Two mutually exclusive events are dependent except they are zero-probabilistic.

Mutual Exclusiveness (Cont'd)

- How we benefit from these two notions?
  - Mutually exclusive $\Rightarrow$ add probabilities to get joint probability
  - Independent $\Rightarrow$ multiply probabilities to get joint pdf.

Extending the notion to three events

- Conditions for 3 events to be independent:
  1. They should be pairwise independent. i.e.,
     - A and B are independent
     - B and C are independent
     - C and A are independent
  2. Knowledge of the joint occurrence of any two events is independent of the third event:

$$\begin{aligned} P(A \cap B|C) &= P(A \cap B) \\ P(B \cap C|A) &= P(B \cap C) \\ P(C \cap A|B) &= P(C \cap A). \end{aligned}$$

Or equivalently, we write $P(A \cap B \cap C) = P(A)P(B)P(C)$ in this case.

An Example

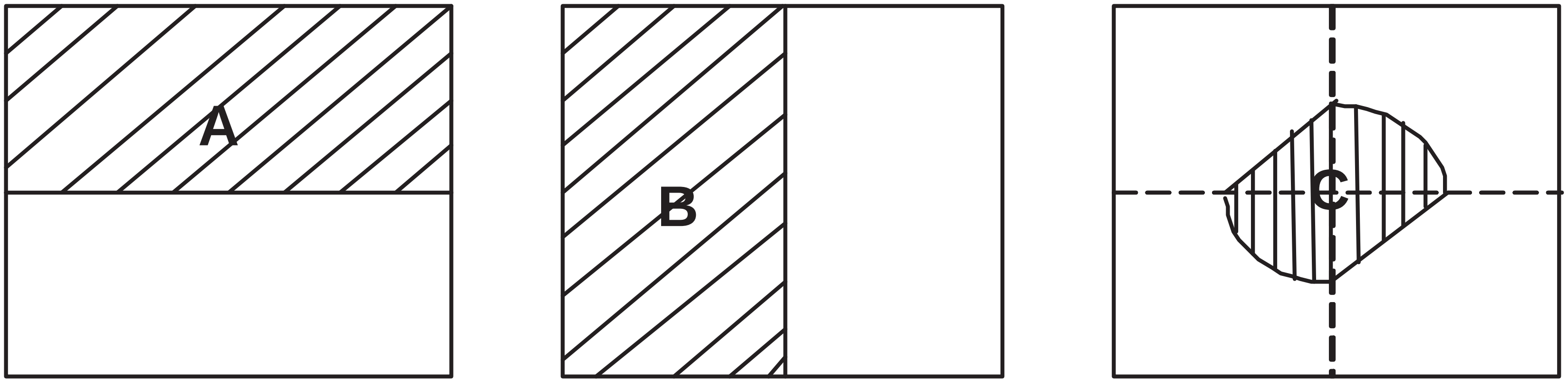


Figure 1: Simple Events

- Consider 3 events A, B and C in the Venn Diagram (Fig. 1).
- Q: Are these 3 events independent?

Example (Cont'd)

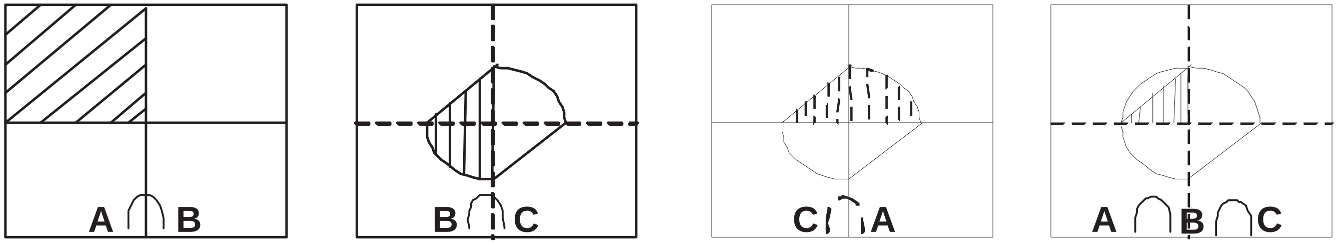


Figure 2: Joint Events

- A: They are pair-wise independent. Since

$$
\begin{aligned}
P(A|B) &= P(A) = \frac{1}{2} \\
P(B|C) &= P(B) = \frac{1}{2} \\
P(C|A) &= P(C) = \frac{1}{2}
\end{aligned}
$$

- However the 2nd condition does not hold:

$$
P(A \cap B|C) \quad < \quad P(A \cap B) = \frac{1}{4}
$$

Generalizing the notion to n events

- Multiplication rule. A set of n events $A_1, A_2, \ldots A_n$ are independent, if the probability of any subset of joint events is equal to the product of their marginal probabilities.

$$P(\cap A_i) = \prod P(A_i)$$

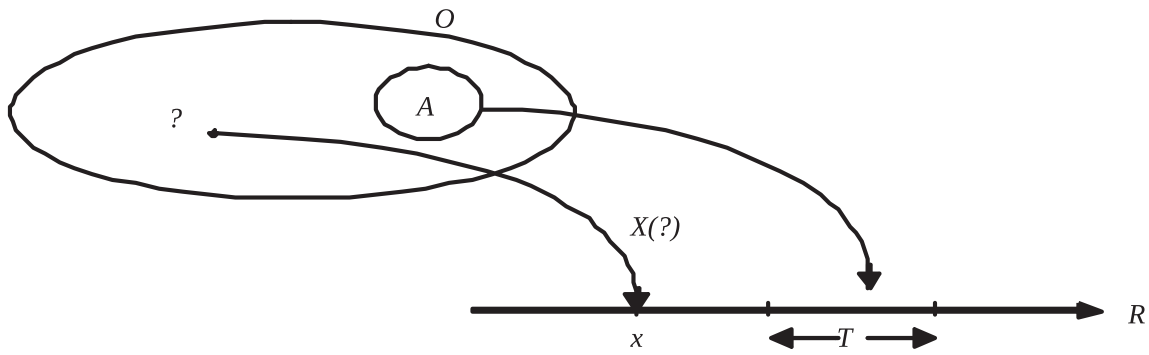
- Equivalently,

$$P(\cap_{i_1}^{i_m} A_i | \cap_{i_{m+1}}^{i_n} A_i) = \prod P(\cap_{i_1}^{i_m} A_i) = \prod P(A_i)$$

- At the heart of the independence, everything is independent of everything else.

- In practice, we assume that the outcomes of separate experiments are all independent.

Random Variables



- A real-valued random variable (rv) is function $X : \Omega \rightarrow \mathbb{R}$ that assigns a value to each outcome $\omega \in \Omega$.
- In the coin toss, suppose we receive \$1 if head appears and pay \$1 otherwise. In this case, we set the rv X to be the amount after first toss:

$$X = \begin{cases} 1, & \text{if H;} \\ -1, & \text{if T.} \end{cases}$$

Joint CDFs

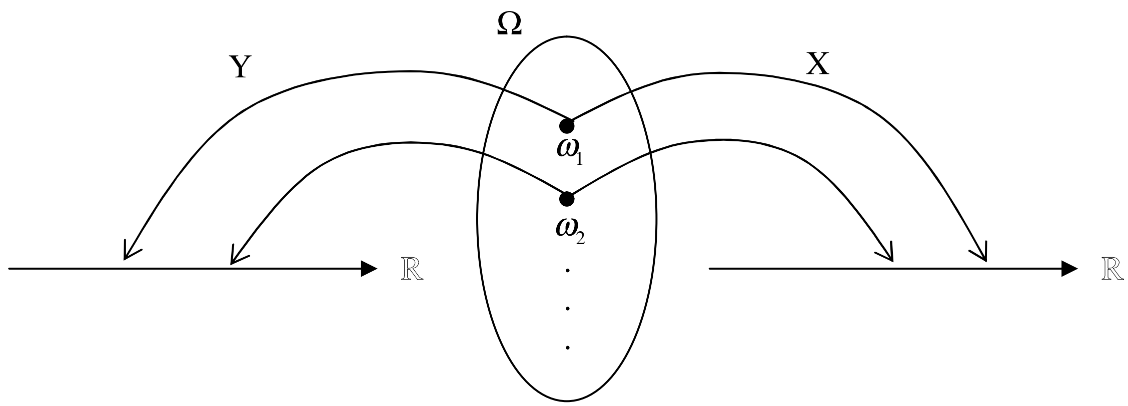


Figure 3: Multivariate RVs

- The joint (cumulative) distribution function of two RVs X and Y is the function $F : \mathbb{R} \rightarrow [0, 1]$ such that

$$
\begin{aligned}
F_{X,Y}(x, y) &= P(X \leq x, Y \leq y) \\
&= P(\omega \in \Omega | \{X(\omega) \leq x\} \cap \{Y(\omega) \leq y\})
\end{aligned}
$$

Independence of Multivariate RVs

- Definition: The two random variables X and Y are independent $\Leftrightarrow$ For any number x and y, the event $A = \{X \leq x\}$ is independent of event $B = \{Y \leq y\}$.

- Recall the joint distribution function of X and Y:

$$F_{X,Y}(x,y) = P[\underbrace{\{X(\omega) \leq x\}}_{A} \cap \underbrace{\{Y(\omega) \leq y\}}_{B}]$$

- But events A and B are independent $\Rightarrow$

$$\begin{aligned} P[A \cap B] &= P(A)P(B) \\ \text{hence } F_{X,Y}(x,y) &= F_X(x)F_Y(y) \quad \forall x, y \in \mathbb{R} \end{aligned}$$

- Differentiating the distribution functions, we get the joint pdf

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

Toy Example Revisited

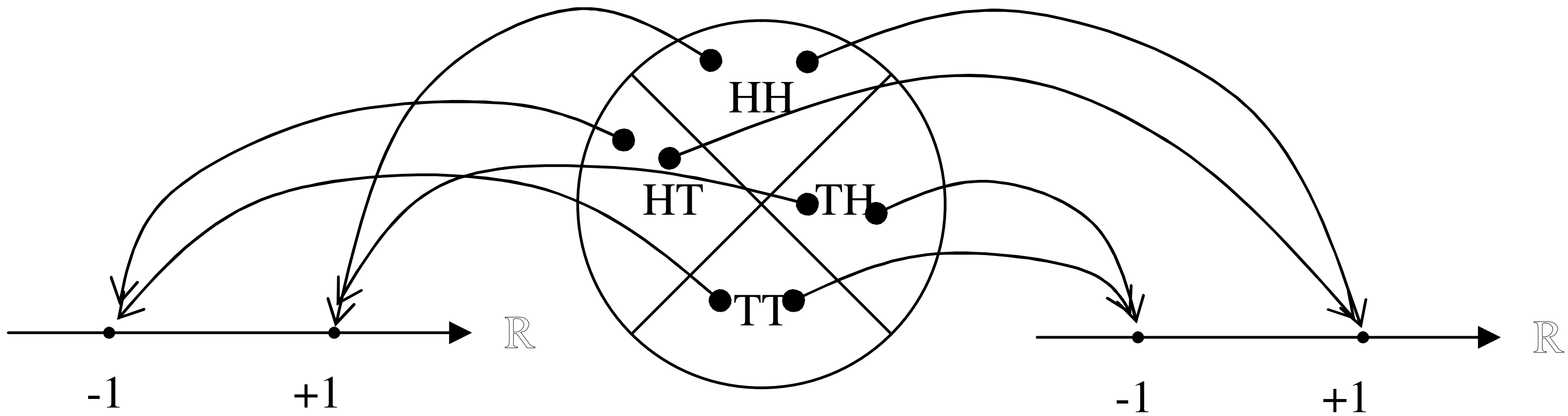


Figure 4: Tossing Coin Twice

- Experiment. Toss a coin twice
- let X and Y be the RVs denoting the outcome of first and second trials, respectively.
- Question: Are X and Y independent RVs ?

Toy Example (Cont'd)

- Solution:

$$
\begin{aligned}
P_X(x) &= \frac{1}{2} \quad x \in \{-1, 1\} \\
P_Y(y) &= \frac{1}{2} \quad y \in \{-1, 1\} \\
P_{X,Y}(x, y) &= \frac{1}{4} \\
&= P_X(x)P_Y(y) \quad \forall\{(x, y)\}.
\end{aligned}
$$

Concluding Remarks

- If RVs X and Y are independent, then
  - $E(XY) = E(X)E(Y)$
  - $\text{var}(X,Y) = 0 \Rightarrow \text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$
  - $f_{Y|X}(y|x) = f_Y(y)$.
- From the 2nd statement, independent $\Rightarrow$ uncorrelated, but not always conversely!
- Generalization. A set of n RVs are independent, if for any finite set of numbers $\{x_1, x_2 \ldots x_n\}$, the events $\{X_1 \leq x_1, X_1 \leq x_1 \ldots X_1 \leq x_1\}$ are independent.
- Equivalently, the joint pdf

$$f_{X_1,\ldots X_n}(x_1 \ldots x_n) \quad = \quad \prod_{i=1}^{n} f_{X_i}(x_i).$$

References

- A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, Hew York, NY, 1991.
- A. Leon-Garcia, *Probability and Random Processes for Electrical Engineering*, Addison-Wesley, 1989.
- R. Yates and D. Goodman *Probability and stochastic processes*, Wiley, 2004.

Thank you!