Q3. Derive the Wiener filter (non-causal) for a stationary process with given spectral characteristics

Background

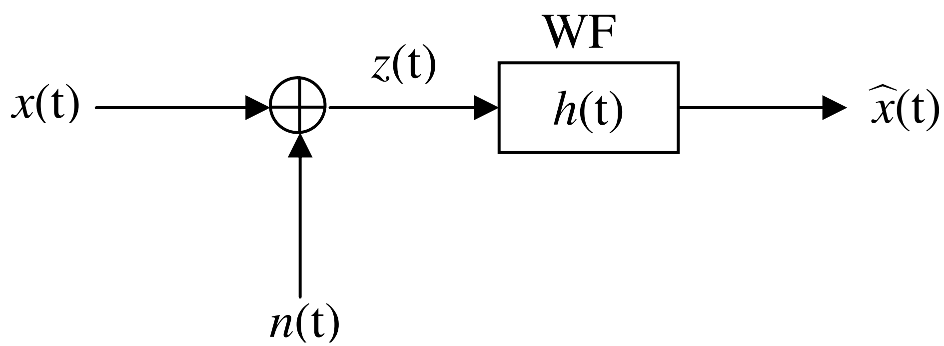

Figure 1: Signal Flow Diagram

- Goal: filter out noise that has corrupted a signal
- Assumptions:
  - Additive noise
  - signal and noise are wide-sense stationary processes
  - Spectral characteristics are known *a priori*.
- Performance criteria: Minimum mean-square error (MMSE).

Model setup

- The input-output relationship of the linear time-invariant system:

$$\begin{aligned}\hat{x}(t) &= \int_{-\infty}^{t} h(\tau)z(t-\tau)d\tau \\ &= h(t) * z(t)\end{aligned}$$

where

- $h(t)$ is the impulse response of the filter
- $z(t)$ is the observed process and related to the unknown signal process $x(t)$ via

$$z(t) = x(t) + n(t)$$

where $n(t)$ is the additive noise.

- $\hat{x}(t)$ is the output process.

Non-causal Wiener Filtering

- Under non-causality, past and future observations are known; hence the Wiener filter(WF) acts as a smoother.
- Redefine the goal to estimate

$$\hat{x}(t) = \mathbb{E}[x(t)|z(\xi), -\infty < \xi < \infty] \tag{1}$$

- The WF minimizes the MSE function:

$$\begin{aligned} J &= \mathbb{E}[\big(x(t) - \hat{x}(t)\big)^2] \\ &= \mathbb{E}[(x(t) - \textstyle\int_{-\infty}^{\infty} h(\alpha) z(t-\alpha) d\alpha)^2] \end{aligned}$$

- Solve the above minimization problem using the orthogonality principle.

Orthogonality Principle

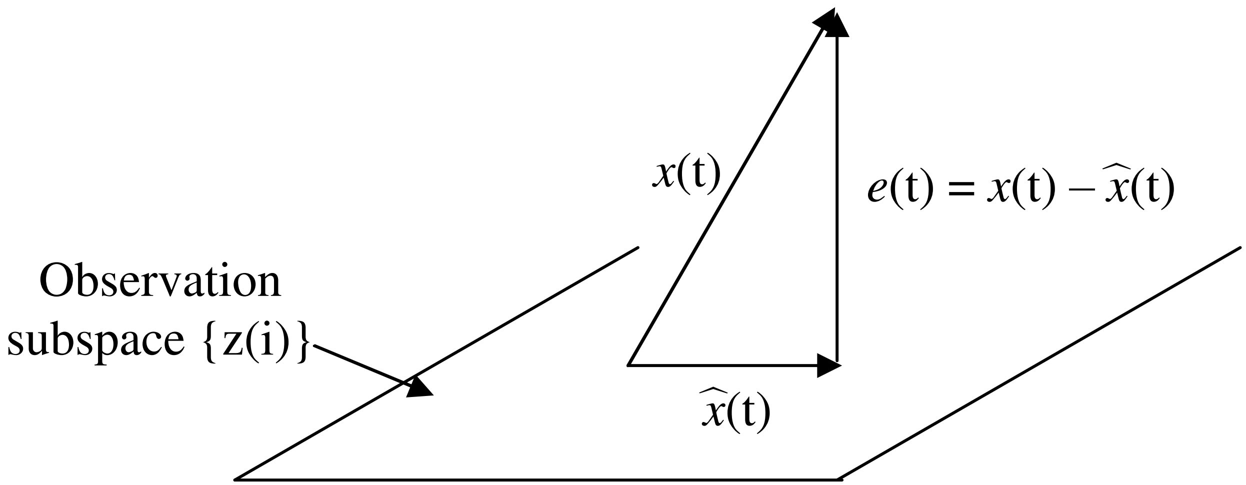

Figure 2: Projection of signal onto the observation subspace

- Idea: To get the minimum error in the MSE sense, the observation subspace has to be orthogonal to the estimation-error subspace.

Orthogonality Principle (Cont'd)

- Using the orthogonality principle we have

$$\begin{aligned} \mathbb{E}\big[(x(t) - \hat{x}(t))z(t-\tau)\big] &= 0 \\ \mathbb{E}\big[\{x(t) - \textstyle\int_{-\infty}^{\infty} h(\alpha)z(t-\alpha)d\alpha\}z(t-\tau)\big] &= 0 \\ \mathbb{E}[x(t)z(t-\tau)] - \textstyle\int_{-\infty}^{\infty} h(\alpha)E[z(t-\alpha)z(t-\tau)]d\alpha &= 0 \\ \Rightarrow R_{xz}(\tau) = \int_{-\infty}^{\infty} h(\alpha)\mathbb{E}[z(t-\alpha)z(t-\tau)]d\alpha. \end{aligned}$$

- Substituting $t = \xi + \tau$ yields

$$\begin{aligned} \text{RHS} &= \int_{-\infty}^{\infty} h(\alpha)\mathbb{E}[z(\xi+\tau-\alpha)z(\xi)]d\alpha \\ &= \int_{-\infty}^{\infty} h(\alpha)R_z(\tau-\alpha)d\alpha \\ &= h(\tau) * R_z(\tau) \end{aligned}$$

- Hence, we get the cross-correlation function:

$$R_{xz}(\tau) = h(\tau) * R_z(\tau)$$

Orthogonality (Cont'd)

- Taking the FT yields the cross-power spectral density:

$$S_{xz}(\omega) = H(\omega)S_z(\omega) \tag{2}$$

- The transfer function of the WF

$$H(\omega) = \frac{S_{xz}(\omega)}{S_z(\omega)} \tag{3}$$

- Further assumptions:
  - $x(t)$ and $n(t)$ are independent stochastic processes $\Rightarrow$ zero cross-correlation.
  - $n(t)$ has zero-mean.
- Under this assumption, we have
  - $R_{xz}(\tau) = R_x(\tau)$
  - $R_z(\tau) = R_x(\tau) + R_n(\tau)$

Orthogonality Principle (Cont'd)

- Equivalently, taking the FT yields
  - $S_{xz}(\omega) = S_x(\omega)$
  - $S_z(\omega) = S_x(\omega) + S_n(\omega)$
- A simplified transfer function of the WF

$$H(\omega) = \frac{S_x(\omega)}{S_x(\omega) + S_n(\omega)} \tag{4}$$

- Interesting Observations:
  - The transfer function is non-zero only where the signal has power content.
  - $H(\omega) = \frac{1}{1+\frac{1}{\frac{S_x(\omega)}{S_n(\omega)}}} \Rightarrow$ emphasizes frequencies where the SNR is large.

Wiener Filter: Block Diagram

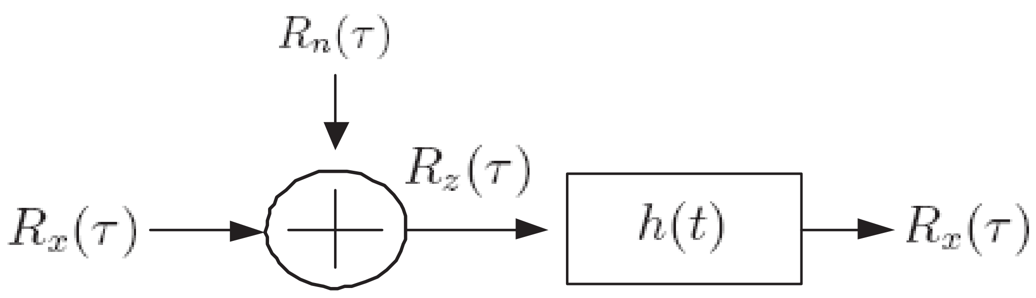

Figure 3: Time Domain

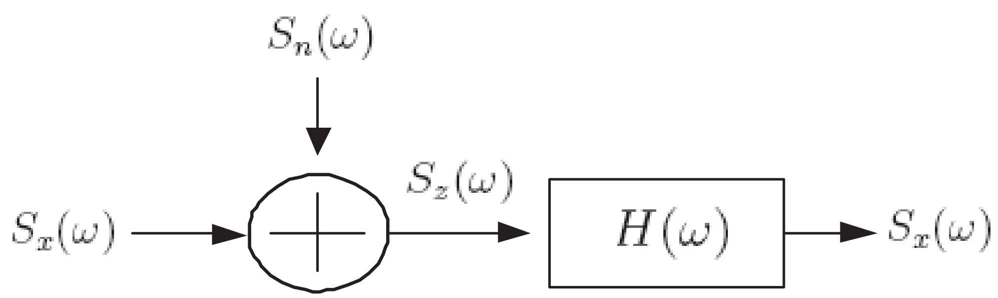

Figure 4: Frequency Domain

Toy Example

- Known. $R_x(\tau) = \frac{\sin(W\tau)}{W\tau}$ with $W = 5 \times 10^3$; $S_n(\omega) = 10^{-5}$
- Find $S_x(\omega)$:

$$S_x(\omega) = \frac{1}{2W} \prod(\frac{2\pi\omega}{W})$$

- Find $H(\omega)$:

$$H(\omega) = \begin{cases} \frac{1}{1.1}, & |\omega| \leq W; \\ 0, & \text{otherwise.} \end{cases}$$

- WF acts as an ideal LPF.

Final Remarks

- WF is applied in image restoration.
- WF has the following limitations:
  - Not amenable to state-vector estimation problems
  - Not applicable to non-stationary signals
  - Non-causal WFs are not suitable for real-time applications

References

- A. Papoulis and S. U. Pillai, *Probability, Random Variables and Stochastic Processes*, 4th ed., McGraw Hill, 2002.
- K. M. Wong, *ECE 762: Detection and Estimation Theory*, Course Notes.
- R. Yates and D. Goodman *Probability and Stochastic Processes*, Wiley, 2004.

Thank you!