Q4. Discuss the linear convolution of the 2-finite length sequences using the DFT
DTFT: A Quick Recap

• Extends the FT for non-periodic discrete-time signals

• Forward DTFT:

\[ X[\Omega] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \]

• Periodic spectrum of period \(2\pi\).

• Abandon to use the DTFT in a digital signal processor for the following reasons:
  - DTFT Spectrum \(X[\Omega]\) is continuous
  - Real signals have finite length
Discrete Fourier Transform

- Extends the DTFT for non-periodic discrete-time signals (finite duration) with discrete frequencies.
- Samples the DTFT spectrum on the interval $[0, 2\pi]$ using $N$ points.
- $N$-point DFT-pairs:
  - Forward
    \[
    X[k] = \sum_{n=0}^{N-1} x[n]W^{kn}, \quad k = 0, \ldots (N - 1)
    \]
    where
    \[
    W = \exp(-j\frac{2\pi}{N}).
    \]
  - Inverse
    \[
    x[n] = \sum_{k=0}^{N-1} X[k]W^{-kn}, \quad n = 0, \ldots (N - 1).
    \]
DFT-pairs in Block-Matrix Form

• Let

\[
x = [x(0), x(1), \ldots x(N - 1)]^T
\]

\[
X = [X(0), X(1), \ldots X(N - 1)]^T
\]

\[
W = \begin{pmatrix}
W^0 & W^0 & \ldots & W^0 \\
W^0 & W^1 & \ldots & W^{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
W^0 & W^{N-1} & \ldots & W^{(N-1)^2}
\end{pmatrix}
\]

• DFT-pairs in a matrix form:

\[
X = Wx \quad (1)
\]

\[
x = \frac{1}{N} W^H x \quad (2)
\]

• Requires \(N^2\) complex multiplications and \(N(N - 1)\) complex additions.
DFT-pairs (Cont’d)

• Taking complex-conjugate of (2) twice replaces the IDFT with DFT:

\[ x = \frac{1}{N} (\text{DFT}(X^*))^*. \]

• Can be implemented using lightening-speed algorithms!
Linear Convolution

- Definition: Suppose two sequences $h[n]$ and $x[n]$ of length $L$ and $P$, respectively.

$$y[n] = (h \ast x)[n] = \sum_{k=-\infty}^{\infty} h[k]x[n - k]$$ (3)

- Basic operations:
  - Time invert one of the sequences
  - Slide it from $-\infty$ to $\infty$
  - When sequences intersect, sum their products

- $y[n]$ is a sequence of length $(L + P - 1)$

- Analogous to computing coefficients of the product of two polynomials.
Example: Linear Convolution
Circular Shift

• Define the circular shift of sequence $x[n]$ of length $N$ as

$$x_1[n] = (\tilde{x}[m - n])\Pi_N(n)$$

where

– $\tilde{x}[n]$ is the periodic extension of $x[n]$

– $\Pi_N(n)$ the rectangular window in the interval $[0, (N - 1)]$.

• 3 basic operations:

  – Periodic extension
  – Normal shift
  – Extraction of the sequence over one period $[0, (N - 1)]$
Example (i): Circular Shift

\( x[n] \)

\( \tilde{x}[n] \)

\( \tilde{x}_1[n] = \tilde{x}[n+2] \)

\( x_1[n] = \begin{cases} 
\tilde{x}[n], & 0 \leq n \leq N-1 \\
0, & \text{otherwise} 
\end{cases} \)
Example (ii): Circular Shift

$$n = 0$$

$$x[0]$$


$$n = 0$$

$$x[2]$$

$$x[3]$$ -- $$x[1]$$ -- $$x[0]$$

$$x[n] = [0, 1, 2, 2]$$

Figure 1: Right Circular Shift on $$x[n] = [0, 1, 2, 2]$$ by 2 points
Circular Convolution

- **Definition**: Suppose two sequences $h[n]$ and $x[n]$ of length $N$ each.

  $$ y[n] = h[n] \otimes x[n] = \left( \sum_{m=0}^{N-1} \tilde{h}[m] \tilde{x}[n-m] \right) \Pi_N(n). $$

- $y[n]$ is a sequence of length $N$.

- **Key Property**:

  $$ h[n] \otimes x[n] \xrightarrow{\text{DFT}} H[k] X[k] $$

- 3 major differences from the linear convolution:
  - Periodic extension
  - Convolution is confined to one period
  - Truncation of one period at the end
Example: Circular Convolution

Periodic the sequences:

\[ x_1[n] \]

\[ x_2[n] \]

Periodic convolution

Get out a period

\[ x_3[n] \]
Zero Padding

Can we perform linear convolution using the DFT? If yes, how?

Extend the length of each sequence such that

\[ N \geq M = (L + P - 1), \]

then

\[ h[n] \otimes x[n] = h[n] * x[n]. \]
Remarks on zero-padding:
- improves the picture of the DTFT
- does not increase spectral resolution or reduce the leakage.
Steps: Linear Convolution via DFT

Figure 2: Flow Diagram

- Choose $N$ to be at least $(L + P - 1)$.
- Pad the two original sequences with zeros to length $N$.
- Compute the $N$-point DFT to obtain $H[k]$ and $X[k]$.
- Compute the point-wise product:

$$Y[k] = H[k]X[k] \quad k = 0, \ldots, (N - 1).$$
Linear Convolution (Cont’d)

• Compute $y[n]$ by taking the $N$-point IDFT of $Y[k]$ as follows:
  – compute the DFT of $Y^*[k]$
  – take the complex conjugate
  – divide by $\frac{1}{N}$

• Save the first $(L + P - 1)$ values of $y[n]$. 
Final Remarks

- To speed up the process, do the followings:
  1. Use FFT in place of DFT with $N$ being some power of 2.
  2. Suppose $h[n]$ is fixed. So pre-compute and save its DFT in advance.

- Linear convolution via DFT is faster than the ‘normal’ linear convolution when

\[
O(N \log(N))_{\text{FFT}} < O(LP)_{\text{normal}}
\]
References

• A. Oppenheim & R. Schafer *Discrete-time signal processing*, 2nd ed.

Thank you!